

Markovian feedback to control continuous variable entanglement

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We present a model to realize quantum feedback control of continuous variable entanglement. It consists of two interacting bosonic modes subject to amplitude damping and achieving entangled Gaussian steady state. The possibility to greatly improve the degree of entanglement by means of Markovian (direct) feedback is then shown.

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Entanglement is one of the most puzzling features of quantum mechanics that has given rise to long debates on foundational aspects since the seminal papers on the subject [1, 2]. Recently, it has been also recognized as a valuable resource for quantum information processing [3]. Thus, its generation and control has become a primary task to accomplish. So far much attention has been devoted to generation and characterization of entanglement, while for its control one usually refers to error correction or distillation procedures [3].

Feedback provides an intuitive way to control a system state, hence entanglement since it is a system state peculiarity. A theory of quantum-limited feedback has been introduced by Wiseman and Milburn [4] leading to relevant experimental achievements [5]. It is based on the direct (immediate) use of the measurement results to alter the system state, hence the name *Markovian feedback*. It turns out to be particularly useful in realizing squeezed states [6]. In fact, monitoring a system's observable will conditionally squeeze its variance. Unconditional squeezing is then obtained by using the measurement results to continuously drive the system into the desired, deterministic, squeezed state. If such arguments are applied to a multiparties system in case of *nonlocal* measurements, the realized squeezing is related to entanglement among the parties [7].

However, the crucial point is to see whether *local* measurements followed by feedback action suffice to control entanglement (because local measurements condition the system to separable states). We shall show that it is indeed possible. By referring to Fig.1, we can consider two interacting subsystems $S1$ and $S2$ each one losing information on its own environment $E1$ and $E2$. Then, monitoring such environments separately will give outcomes' currents that can be joined and exploited for feedback action.

Recent results on finite dimensional systems show that such feedback scheme is helpful in enforcing entanglement [8]. However, the original gedanken experiment proposal of Ref.[2] refers to infinite dimensional systems; moreover continuous variables offer a lot of advantages in information processing [9]. We henceforth focus our attention on the possibility to control the degree of en-

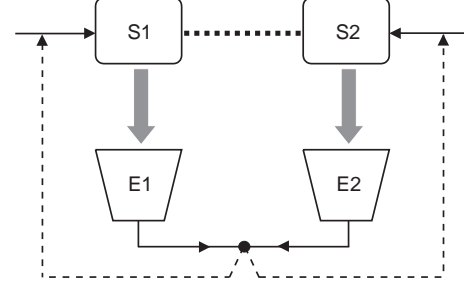


FIG. 1: The studied scheme. Two subsystems $S1$ and $S2$ interact (thick dashed line). Each one is coherently driven (left-right arrow and right-left arrow) and loses information in its own environment, $E1$ and $E2$. The environments are separately monitored and then a joint current is exploited for a feedback action on the driving fields (dashed lines).

tanglement for a two-party quantum Gaussian state. A paradigm of the model of Ref.[2] is represented by a non-degenerate parametric oscillator [10] where two bosonic modes a_1 and a_2 interact through a quadratic Hamiltonian of the type ($\hbar = 1$)

$$H_{int} = i\chi (a_1^\dagger a_2^\dagger - a_1 a_2), \quad (1)$$

with $\chi \in \mathbb{R}$ the coupling strength. In addition we consider the two modes driven by a Hamiltonian

$$H_{drive} = \alpha (a_1^\dagger + a_2^\dagger) + \alpha^* (a_1 + a_2), \quad (2)$$

where $\alpha \in \mathbb{C}$ is the driving amplitude (assumed equal for both modes). Hence, the total Hamiltonian results

$$H_{tot} = H_{int} + H_{drive}. \quad (3)$$

Now, suppose that the system loses information on each mode through an amplitude damping channel at rate κ (hereafter we consider $\kappa = 1$ so that all physical quantities become dimensionless). Then, by introducing the modes quadratures

$$X_j = \frac{a_j + a_j^\dagger}{\sqrt{2}}, \quad Y_j = \frac{a_j - a_j^\dagger}{i\sqrt{2}}, \quad j = 1, 2, \quad (4)$$

the system dynamics can easily be described by the quantum Langevin equations [11]

$$\dot{X}_1 = \chi X_2 - i \frac{\alpha - \alpha^*}{\sqrt{2}} - \frac{1}{2} X_1 + \mathcal{X}_1, \quad (5)$$

$$\dot{Y}_1 = -\chi Y_2 - \frac{\alpha + \alpha^*}{\sqrt{2}} - \frac{1}{2} Y_1 + \mathcal{Y}_1, \quad (6)$$

$$\dot{X}_2 = \chi X_1 - i \frac{\alpha - \alpha^*}{\sqrt{2}} - \frac{1}{2} X_2 + \mathcal{X}_2, \quad (7)$$

$$\dot{Y}_2 = -\chi Y_1 - \frac{\alpha + \alpha^*}{\sqrt{2}} - \frac{1}{2} Y_2 + \mathcal{Y}_2, \quad (8)$$

where \mathcal{X}_j and \mathcal{Y}_j ($j = 1, 2$) are hermitian operators representing the vacuum noise entering the system. They have equal time correlations

$$\langle \mathcal{X}_j \mathcal{X}_k \rangle = \langle \mathcal{Y}_j \mathcal{Y}_k \rangle = \frac{1}{2} \delta_{jk}, \quad (9)$$

$$\langle \mathcal{X}_j \mathcal{Y}_k \rangle = -\langle \mathcal{Y}_j \mathcal{X}_k \rangle = i \frac{1}{2} \delta_{jk}, \quad j, k = 1, 2. \quad (10)$$

By monitoring each mode environment it would be possible to measure a quadrature for each subsystem, e.g. by using homodyne detection [12]. Suppose this is done with an overall efficiency $\eta \in [0, 1]$ (i.e., η accounts for the detectors efficiency, the fraction of the field being measured, etc.), then we can write local currents like [11]

$$I_j(t) = \eta X_j + \sqrt{\eta} \mathcal{W}_j, \quad j = 1, 2, \quad (11)$$

where \mathcal{W}_j ($j = 1, 2$) are hermitian operators representing the vacuum noise affecting the currents. They have equal time correlations

$$\langle \mathcal{W}_j \mathcal{W}_k \rangle = \frac{1}{2} \delta_{jk}, \quad j, k = 1, 2. \quad (12)$$

We can now combine the currents I_j to get the following joint current

$$I(t) = \eta (X_1 - X_2) + \sqrt{\eta} (\mathcal{W}_1 - \mathcal{W}_2), \quad (13)$$

through which it would be possible to gain information about the quantity $X_1 - X_2$. Its variance in absence of environmental effects and for increasing value of χ , tends to zero [10], corresponding to the maximally entangled state of the type discussed in Ref.[2]. Thus, the fact that it goes below 1 (vacuum fluctuations) can be roughly considered as signature of entanglement. In such a sense, we may assume the current in Eq.(13) giving us information about the system state entanglement.

The steady state solution of Eqs.(5)-(8) results

$$\langle X_1 \rangle_{ss} = \langle X_2 \rangle_{ss} = -i\sqrt{2} \frac{\alpha - \alpha^*}{1 - 2\chi}, \quad (14)$$

$$\langle Y_1 \rangle_{ss} = \langle Y_2 \rangle_{ss} = -\sqrt{2} \frac{\alpha + \alpha^*}{1 + 2\chi}, \quad (15)$$

which is stable for $\chi < 1/2$.

Subtracting Eq.(7) from (5) we get a quantum Langevin equation resembling a Ornstein-Uhlenbeck process [13]. It can be easily solved to get the steady state variance

$$\langle (X_1 - X_2)^2 \rangle - (\langle X_1 \rangle_{ss} - \langle X_2 \rangle_{ss})^2 = \frac{1}{1 + 2\chi}. \quad (16)$$

We see that the variance (16) goes below 1 (vacuum fluctuations) as soon as $\chi > 0$, thus we infer the presence of entanglement in the steady state. However, since it must be $\chi < 1/2$, the variance (16) is limited from below by $1/2$. By converse, the amount of entanglement will be limited. That is, the damping channel degrades the system state preventing it to become maximally entangled like that of Ref.[2].

We can now think to control such a process, hence to control entanglement, by using feedback action accordingly to the information gained about $X_1 - X_2$. To this end we consider an additional Hamiltonian proportional to the current (13) and driving the operator $Y_1 - Y_2$,

$$H_{fb} = \frac{\lambda}{\eta} I(t - \tau) (Y_1 - Y_2). \quad (17)$$

Here λ represents the feedback strength and τ the feedback loop delay time. The above choice of the driving is motivated by the fact that we want to affect the variance of $X_1 - X_2$. The feedback action (17) can be realized by a modulation of the driving fields accordingly to the current I .

Adding the Hamiltonian (17) to the Langevin equations (5)-(8) means to add, for a generic operator \mathcal{O} , the term

$$\dot{\mathcal{O}}_{fb} = \frac{i}{\eta} \int_0^t d\tau G(\tau) I(t - \tau) \lambda [Y_1 - Y_2, \mathcal{O}], \quad (18)$$

where G is the feedback response function [4]. Practically, it accounts for the feedback loop delay time, however hereafter we assume it negligibly small, hence G can be considered as a Dirac delta function.

When inserting Eq.(18) into Eqs.(5)-(8), one has to also account for the noise carried inside the system by the current (13). In presence of imperfect detection the current may be non trivially related with the input noise. The following correlations hold at equal times [14]

$$\langle \mathcal{W}_j \mathcal{X}_k \rangle = \langle \mathcal{X}_j \mathcal{W}_k \rangle = \frac{\sqrt{\eta}}{2} \delta_{jk}, \quad (19)$$

$$\langle \mathcal{W}_j \mathcal{Y}_k \rangle = -\langle \mathcal{Y}_j \mathcal{W}_k \rangle = i \frac{\sqrt{\eta}}{2} \delta_{jk}, \quad j, k = 1, 2. \quad (20)$$

It is immediate to see that in the perfect efficiency case ($\eta = 1$) one can identify the feedback noise with the input noise, while in the opposite case ($\eta = 0$) the two noises are uncorrelated (as it can be easily expected since in such a case the feedback noise has nothing to do with the vacuum input noise).

It is worth noting that the feedback action (18) does not affect the steady state values (14)-(15). Then we can rewrite Eqs.(5)-(8) in presence of feedback, for only the steady state fluctuations

$$x_j = X_j - \langle X_j \rangle_{ss}, \quad (21)$$

$$y_j = Y_j - \langle Y_j \rangle_{ss}, \quad j = 1, 2. \quad (22)$$

Let us introduce the operator vectors

$$v \equiv \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix}, \quad n \equiv \begin{pmatrix} \mathcal{X}_1 + \frac{\lambda}{\sqrt{\eta}}(\mathcal{W}_1 - \mathcal{W}_2) \\ \mathcal{Y}_1 \\ \mathcal{X}_2 - \frac{\lambda}{\sqrt{\eta}}(\mathcal{W}_1 - \mathcal{W}_2) \\ \mathcal{Y}_2 \end{pmatrix}, \quad (23)$$

then, the feedback modified Langevin equations read, in a compact way,

$$\dot{v} = -Mv + n, \quad (24)$$

where

$$M = \begin{pmatrix} \frac{1}{2} - \lambda & 0 & \lambda - \chi & 0 \\ 0 & \frac{1}{2} & 0 & \chi \\ \lambda - \chi & 0 & \frac{1}{2} - \lambda & 0 \\ 0 & \chi & 0 & \frac{1}{2} \end{pmatrix}. \quad (25)$$

The symmetric noise correlation matrix is

$$\begin{aligned} N &\equiv \frac{1}{2} (\langle nn^T \rangle + \langle nn^T \rangle^T) \\ &= \begin{pmatrix} \frac{1}{2} + \lambda + \frac{\lambda^2}{\eta} & 0 & -\lambda - \frac{\lambda^2}{\eta} & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ -\lambda - \frac{\lambda^2}{\eta} & 0 & \frac{1}{2} + \lambda + \frac{\lambda^2}{\eta} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}, \end{aligned} \quad (26)$$

therefore the stationary symmetric correlation matrix

$$\Gamma \equiv \frac{1}{2} (\langle vv^T \rangle + \langle vv^T \rangle^T), \quad (27)$$

can be derived from the typical relation for Ornstein-Uhlenbeck-like processes [13]

$$M\Gamma + \Gamma M^T = N. \quad (28)$$

For the following purposes, we write Γ in terms of its 2×2 submatrices

$$\Gamma = \begin{pmatrix} \gamma & \sigma \\ \sigma^T & \gamma \end{pmatrix}, \quad (29)$$

where the matrix elements, by virtue of Eq.(28), result

$$\gamma_{11} = \frac{(1/2) - \lambda + (1 - 2\chi)(\lambda + \lambda^2/\eta)}{1 - 4\lambda + 8\lambda\chi - 4\chi^2}, \quad (30)$$

$$\gamma_{12} = \gamma_{21} = 0, \quad (31)$$

$$\gamma_{22} = \frac{1}{2} \left(\frac{1}{1 - 4\chi^2} \right), \quad (32)$$

and

$$\sigma_{11} = \frac{\chi - \lambda - (1 - 2\chi)(\lambda + \lambda^2/\eta)}{1 - 4\lambda + 8\lambda\chi - 4\chi^2}, \quad (33)$$

$$\sigma_{12} = \sigma_{21} = 0, \quad (34)$$

$$\sigma_{22} = -\frac{\chi}{1 - 4\chi^2}. \quad (35)$$

Since the steady state is Gaussian it is completely characterized by the correlation matrix (27). Then, its degree of entanglement can be quantified by means of the logarithmic negativity [15]

$$L \equiv \begin{cases} -\log(2\zeta) & \text{if } \zeta < 1 \\ 0 & \text{otherwise} \end{cases}, \quad (36)$$

with

$$\zeta \equiv \sqrt{(\det \gamma - \det \sigma) - \sqrt{(\det \gamma - \det \sigma)^2 - \det \Gamma}}. \quad (37)$$

We have numerically evaluated the quantity

$$L_{fb} \equiv \max_{\lambda \in \mathbb{R}} L, \quad (38)$$

by using Eqs.(30)-(35). In doing so we have accounted for the stability condition $M \geq 0$ and for the Heisenberg uncertainty condition $\Gamma + (i/2)\Omega \geq 0$ [16]. Here Ω is the standard symplectic form

$$\Omega \equiv \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}, \quad \omega \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (39)$$

As matter of fact, not all real values of λ are admitted since the addition of feedback may lead to unstable or unphysical states. Thus, the maximization (38) is performed on a ‘valid’ range of λ ’s values. The extension of such a range clearly enlarges as η decreases. Anyway, the maximum is always achieved for values $-(1/2) < \lambda < 0$.

The quantity L_{fb} of Eq.(38) is shown in Fig.2 as function of χ for different values of η . Notice that L_{fb} for $\eta = 0$ (lower curve) corresponds to the case of no feedback. It tells us that in absence of feedback, entanglement arises for $\chi > 0$ and increases with it till the value $L = 1$. Instead, much higher values can be obtained by the feedback action, even with low values of η . The benefit of feedback increases by increasing η and it is particularly manifest for high values of χ . In the limit case of $\eta \rightarrow 1$ by approaching the instability $\chi \rightarrow 1/2$ entanglement increases indefinitely. This is because in such a case the feedback is able to completely recycle the information lost by the system into environment, i.e. it somehow suppresses the amplitude damping, making a maximally entangled state achievable [2, 10].

It is worth noting that the presented scheme can be implemented by using an experimental set up similar to that of Ref.[17], where the two fields emerging from an optical parametric oscillator were separately subjected to

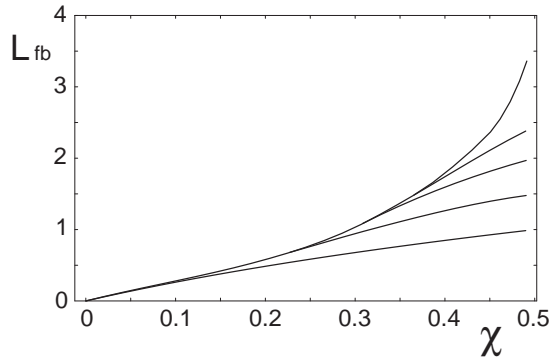


FIG. 2: Logarithmic negativity L_{fb} as function of χ . Curves from bottom to top are for $\eta = 0, 0.3, 0.5, 0.7, 0.99$.

homodyne measurements and the relative currents combined. Then one has to employ driving field modulators to realize the feedback action. Any delays in classical communication (including the feedback loop) must be much smaller than the typical time scale of the system (κ^{-1}) and much smaller than the inverse of relevant bandwidth. Experimentalists can certainly achieve loop delays smaller than 10^{-8} s [18] which already satisfies the above requirements for set ups similar to [17]. From this point of view Markovian (direct) feedback has also the advantage of not requiring information processing after measurement. To give a numerical example, in Ref.[17] an overall efficiency of ~ 0.7 was achieved, which already means the possibility of $\sim 250\%$ improvement of entanglement close to the threshold (see Fig.2).

The feedback action results a Gaussian operation from the quadratic form of Eq.(17) together with Eq.(13). Moreover, the presence of vacuum noises \mathcal{W}_j in such equations legitimate us to intend the feedback action as local operation supplemented by classical communication (LOCC). As matter of fact, in absence of interaction ($\chi = 0$) the feedback action is unable to generate entanglement as it is evident from Fig.2. However, in Refs. [19] it was shown the impossibility to enhance (distill) entanglement by means of Gaussian LOCC. The key point is that, in contrast with Refs. [19], we have considered continuous measurement and continuous communication between the two parties, including the exchange of noise of quantum origin. And the correlations among \mathcal{W}_j and $\mathcal{X}_j, \mathcal{Y}_j$ are able to reduce the quantum noisy effects on the system, hence to enforce the interaction between the subsystems. This corresponds to an overall nonseparable Gaussian map in the sense of Ref. [19], though the real physical operations are LOCC. Thus, the presented approach may shed further light on the subject of entanglement distillation.

It is also to remark that the our model extends behind Eq.(1) to any other quadratic interaction, because Gaussian state's entanglement is always related to the squeez-

ing of a suitable quadrature combination [20], which then might play the role of $X_1 - X_2$.

In conclusion, we have shown the possibility to greatly improve the steady state entanglement in an open quantum system by using a feedback action. In the studied case, where Gaussian states were involved, Markovian (direct) feedback suffices to reach the goal, while we guess that other feedback procedures, like state estimation based feedback [21], could be more powerful in other contexts.

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